

# Optimal Extraction and Taxation of Strategic Natural Resources: A Differential Game Approach

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## Abstract

This paper studies the optimal extraction and taxation of nonrenewable natural resources. It is well known the market values of the main strategic resources such as oil, natural gas, uranium, copper,...etc, fluctuate randomly following global and seasonal macro-economic parameters, these values are modeled using Markov switching Lévy processes. We formulate this problem as a differential game where the two players are the mining company whose aim is to maximize the revenues generated from its extracting activities and the government agency in charge of regulating and taxing natural resources. We prove the existence of a Nash equilibrium and characterize the value functions of this differential game as the unique viscosity solutions of the corresponding Hamilton Jacobi Isaacs equations. Furthermore, optimal extraction and taxation policies that should be applied when the equilibrium is reached are derived. In addition, we construct and prove the convergence of a numerical scheme for approximating the value functions and optimal policies. A numerical example is presented to illustrate our findings.

**Keywords:** Natural Resource Economics, Lévy Processes, Stochastic Differential Games, Markov Switching, Viscosity Solutions.

## 1 Introduction

Natural resources have always been the main sources of income for some developing countries. The extraction of natural resources in developing countries is usually done by large multinational corporations. The revenues generated from the sales of these resources in world markets as well as the taxes those countries levy on multinational mining companies accounted for more than half of the budget of those resource-rich developing countries. Thus, the production and regulation of strategic natural resources have always been one of the prime topics of discussion in political and scientific circles.

The earliest scientific contribution on the extraction of natural resources was done by Hotelling (1931), he derived an optimal extraction policy under the assumption that the commodity price is constant. A wide range of economists have extended the Hotelling model by taking into account the uncertainty when modeling commodity prices. Among many others, one can cite the work of Sweeney (1977), Hanson (1980), Solow and Wan (1976), Pindyck (1978), (1980), Sweeney (1977) for various extensions of the basic Hotelling model. Cherian *et al.* (1998) studied the optimal extraction of nonrenewable resources as a stochastic optimal control problem with two state variables, the commodity price and the size of the remaining reserve. They solved the control problem numerically by using Markov chain approximation methods. Recently Aleksandrov *et al.* (2012) studied the optimal production of oil as an American-style real option and used Monte-Carlo methods to approximate the optimal production rate when the oil price follows a mean-reverting process.

The taxation of natural resources has also generated a great deal of interest in the academic literature. One

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can cite the work of Beals *et al.* (1980) on tax and investment policies for hard minerals and the contributions of Bradley *et al.* (1981), Heaps and Helliwell (1985), and Bhattacharyya (1998) on efficient tax policies for natural resources and energy.

The extraction and taxation of natural resources are in fact two sides of the same coin when it comes to generating revenues for the public finances of resource-rich developing countries. However, throughout the scientific literature these issues have usually been treated separately. The main contribution of this work is that we treat these problems in their natural setting by highlighting the interplay between the extraction and taxation policies of strategic resources. We use the framework of noncooperative differential games to tackle these issues. We formulate this problem as a differential game where the two players are the multinational mining company and the government. Obviously the multinational mine company wants to maximize its share of profits from the extraction activities and the government wants to maximize its share of profits from the sale of the extracted resource in world market as well as the income tax it levies on the multinational company. To the best of our knowledge, this is the first time this approach is used to characterize the interplay between extraction and taxation of natural resources. It is also self evident that commodity prices in exchange markets fluctuate following various macro economical and global geopolitical forces. It is therefore crucial to take into account the random dynamic of the commodity value when solving the optimal extraction and taxation problems in order to maintain the validity of the result obtained in the procedure. In this paper, we use regime switching Lévy processes to model commodity prices. Regime switching models have been extensively used in the financial economics literature since its introduction by Hamilton (1989). Many authors have studied the control of systems that involve regime switching using a hidden Markov chain, one can cite Zhang and Yin (1998), (2005), Pemy and Zhang (2006), Pemy (2011), (2014) among others.

The paper is organized as follows. In the next section, we formulate the problem under consideration. In Section 3, we proof the existence of the Nash equilibrium. In section 4 we derive the properties of the value functions and characterize them as unique viscosity solutions of the Hamilton Jacobi Isaacs equations. And in section 5, we construct a finite difference approximation scheme and prove its convergence to the value functions. Finally, in section 6, we give a numerical example.

## 2 Problem formulation

Consider a company which has a Production Sharing Agreement with the government of a country rich in natural resources. The agreement is for the extraction of a nonrenewable natural resource. Both parties will share the profits from the sales of the mineral on world markets following a simple rule where the company takes  $\theta$  percent and the government  $1 - \theta$  percent of the profits. We assume that the mining lease has expiration  $0 < T < \infty$  and the value of nonrenewable resource at time  $t$  is  $X_t$ . Given that commodity values are very sensitive global macro-economical and geopolitical shocks, we model  $X_t$  as a regime switching Lévy process with two states. Let  $\alpha(t) \in \mathcal{M} = \{1, 2\}$  be a finite state Markov chain that captures the state of the commodity marketplace:  $\alpha(t) = 1$  indicates the bull market at time  $t$  and  $\alpha(t) = 2$  represents a bear market at time  $t$ . The generator of this Markov chain is

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}, \quad \lambda_1 > 0, \lambda_2 > 0.$$

Let  $(\eta_t)_t$  be a Lévy process and let  $N$  be the Poisson random measure of  $(\eta_t)_t$ ,  $N(t, U) = \sum_{0 < s \leq t} \mathbf{1}_U(\eta_s - \eta_{s-})$  for any Borel set  $U \subset \mathbb{R}$ . Moreover, let  $\nu$  be the Lévy measure of  $(\eta_t)_t$  we have  $\nu(U) = \bar{E}[N(1, U)]$  for any Borel set  $U \subset \mathbb{R}$ . The differential form of  $N$  is denoted by  $N(dt, dz)$ , we define the differential

$\bar{N}(dt, dz)$  as follows

$$\bar{N}(dt, dz) = \begin{cases} N(dt, dz) - \nu(dz)dt & \text{if } |z| < 1 \\ N(dt, dz) & \text{if } |z| \geq 1. \end{cases}$$

We assume that the Lévy measure  $\nu$  has finite intensity,

$$\Gamma = \int_{\mathbb{R}} \nu(dz) < \infty. \quad (1)$$

In other words, the total sum of jumps and spikes of the commodity price during the lifetime of the contract is finite. Let  $K < \infty$  be the total size of the mine at the beginning of the contract, let  $Y(t)$  be the size of the remaining resource in the mine yet to be extracted by time  $t$ ,  $Y(t) \in [0, K]$ . We model the evolution of the profit sharing agreement as a differential game where the two players are the mining company and the government. Each player acting as a controller will to maximize its own profit throughout the duration of the contract. The mining company will try to maximize its share of profits from the sales of mineral in world markets, while the government will also try to maximize both its share of the profits from the sales of mineral in world markets and the income tax it levies on the mining company. We denote the mining company as Player 1 and the government as Player 2. We assume that the processes  $X(t), Y(t)$  follow the dynamical system

$$\left\{ \begin{array}{l} dX(t) = \left( \mu(t, X(t), u_1(t), u_2(t), \alpha(t))dt \right. \\ \quad \left. + \sigma(t, X(t), \alpha(t))dW(t) \right. \\ \quad \left. + \int_{\mathbb{R}} \gamma(t, X(t), \alpha(t), z) \bar{N}(dt, dz) \right), \\ dY(t) = -u_1(t)dt, \\ X(s) = x \geq 0, \quad Y(s) = y \geq 0, \quad 0 \leq s \leq t \leq T, \end{array} \right. \quad (2)$$

where  $u_1(t) \in U_1 = [0, \bar{u}_1]$  is the extraction rate chosen that company and  $u_2(t) \in U_2 = [0, \bar{u}_2]$  is the tax rate chosen by the government. The processes  $u_1(t), u_2(t)$  are control variables, and  $W(t)$  is the Wiener process defined on a probability space  $(\Omega, \mathcal{F}, P)$ . Moreover, we assume that  $W(t), \eta_t$  and  $\alpha(t)$  are independent.

The functions  $\mu : [0, T] \times \mathbb{R} \times [0, \bar{u}_1] \times [0, \bar{u}_2] \times \mathcal{M} \rightarrow \mathbb{R}$ ,  $\sigma : [0, T] \times \mathbb{R} \times \mathcal{M} \rightarrow \mathbb{R}$  and  $\gamma : [0, T] \times \mathbb{R} \times \mathcal{M} \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the following properties:

- **Lipschitz continuity:** There exists a constant  $C > 0$  such that, for all  $t, x, y, u_1, u_2$  we have

$$\begin{aligned} & |\mu(t, x, u_1, u_2, i) - \mu(t, y, u_1, u_2, i)|^2 \\ & + |\sigma(t, x, i) - \sigma(t, y, i)|^2 \\ & + \int_{|z| < 1} |\gamma(t, x, i, z) - \gamma(t, y, i, z)|^2 \nu(dz) \\ & < C|x - y|^2. \end{aligned} \quad (3)$$

- **Growth condition:** There exists a constant  $C > 0$  such that, for all  $t, x, y, u_1, u_2$  we have

$$\begin{aligned} & |\mu(t, x, u_1, u_2, i)|^2 + \|\sigma(t, x, i)\|^2 + \\ & \int_{|z| < 1} |\gamma(t, x, i, z)|^2 \nu(dz) < C(1 + |x|^2). \end{aligned} \quad (4)$$

The assumptions (3) and (4) guarantee that for any Lebesgue measurable control  $u_1(\cdot)$  and  $u_2(\cdot)$ , taking values on compact sets  $[0, \bar{u}_1]$  and  $[0, \bar{u}_2]$  respectively, the equation (2) has a unique solution, such control processes will be called admissible control. For more, one can refer to Øksendal and Sulem (2004). For each initial data  $(s, x, y, i)$  we denote by  $\mathcal{U}_j(s, x, y, i)$  the set of admissible controls which is just the set of all controls  $u_j(\cdot)$  taking values in  $U_j = [0, \bar{u}_j]$  such that  $X(s) = x$ ,  $Y(s) = y$ , and which are  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted where  $\mathcal{F}_t = \sigma\{\alpha(\xi), W(\xi); \xi \leq t\}$ ,  $j = 1, 2$ .

Let  $L_1(t, x, y, u_1, u_2, i)$  and  $L_2(t, x, y, u_1, u_2, i)$  be respectively the running profit of the mining company and the government. These functions are defined on  $[0, T] \times \mathbb{R} \times [0, K] \times [0, \bar{u}_1] \times [0, \bar{u}_2] \times \mathcal{M}$ . Let  $\Phi_1(t, x, y, i)$  and  $\Phi_2(t, x, y, i)$  be respectively the final profit of the company and the government when the contract ends. The functions  $L_1, L_2, \Phi_1$  and  $\Phi_2$  are at least Lipschitz continuous in  $t, x$  and  $y$ . Given a risk-free rate  $r > 0$ , the payoff of each player  $i = 1, 2$  is defined as follows;

$$\begin{aligned} & J_i(s, x, y, \iota; u_1, u_2) \\ &= E \left[ \int_s^T e^{-r(t-s)} L_i(t, X(t), Y(t), u_1(t), u_2(t), \alpha(t)) dt \right. \\ & \quad \left. + e^{-r(T-s)} \Phi_i(T, X(T), Y(T), \alpha(T)) \right] \Bigg| X(s) = x, \\ & \quad Y(s) = y, \alpha(s) = \iota \Bigg]. \end{aligned}$$

It is obvious each player want to maximize its own payoff. The company will try to maximize its payoff by adjusting the extraction rate  $u_1(\cdot)$ , while the government will maximize its payoff by changing the tax rate  $u_2(\cdot)$  following the changes in the commodity price  $X(t)$ . This is therefore the setting of a noncooperative game. Our goal is to find a noncooperative Nash equilibrium  $(u_1^*, u_2^*)$  such that

$$J_1(s, x, y, \iota; u_1^*, u_2^*) \geq J_1(s, x, y, \iota; u_1, u_2^*), \quad (5)$$

for all  $u_1(\cdot) \in \mathcal{U}_1(s, x, y, \iota)$ ,

$$J_2(s, x, y, \iota; u_1^*, u_2^*) \geq J_2(s, x, y, \iota; u_1^*, u_2), \quad (6)$$

for all  $u_2(\cdot) \in \mathcal{U}_2(s, x, y, \iota)$ .

### 3 Nash Equilibrium

**Definition 3.1** Let  $(u_1^*, u_2^*)$  be a Nash equilibrium of our differential game, the functions

$$\begin{aligned} V_1(s, x, y, \iota) &= \sup_{u_1 \in \mathcal{U}_1} J_1(s, x, y, \iota; u_1, u_2^*) \\ V_2(s, x, y, \iota) &= \sup_{u_2 \in \mathcal{U}_2} J_2(s, x, y, \iota; u_1^*, u_2) \end{aligned}$$

are called value function of Player 1 and 2 respectively.

In order to find the optimal strategies  $u_1^*, u_2^*$  of the Nash equilibrium we first have to derive the value functions  $V_1, V_2$  of the differential game, then derive the optimal strategies. Formally the value functions  $V_1, V_2$  should satisfy the following Hamilton Jacobi Isaacs equations. Assuming that we have a Nash equilibrium  $u_1^*, u_2^*$  let us define corresponding Hamiltonians:

$$\begin{aligned} & H_1(s, x, y, \iota, V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial x^2}) \\ &= rV - \sup_{u_1 \in U_1} \left( \frac{1}{2} \sigma^2(s, x, y, \iota) \frac{\partial^2 V}{\partial x^2} \right. \end{aligned}$$

$$\begin{aligned}
& +\mu(s, x, u_1, u_2^*, \iota) \frac{\partial V}{\partial x} - u_1 \frac{\partial V}{\partial y} \\
& + \int_{\mathbb{R}} \left( V(s, x + \gamma(s, x, \iota, z), y, \iota) - V(s, x, y, \iota) \right. \\
& \left. - \mathbf{1}_{\{|z|<1\}}(z) \frac{\partial V(s, x, y, \iota)}{\partial x} \cdot \gamma(s, x, \iota, z) \right) \nu(dz) \\
& + L_1(s, x, y, u_1, u_2^*, \iota) + QV(s, x, y, \cdot)(\iota) \Big), \tag{7}
\end{aligned}$$

and

$$\begin{aligned}
& H_2(s, x, y, \iota, V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial x^2}) \\
& = rV - \sup_{u_2 \in U_2} \left( \frac{1}{2} \sigma^2(s, x, y, \iota) \frac{\partial^2 V}{\partial x^2} \right. \\
& + \mu(s, x, u_1^*, u_2, \iota) \frac{\partial V}{\partial x} - u_1 \frac{\partial V}{\partial y} \\
& + \int_{\mathbb{R}} \left( V(s, x + \gamma(s, x, \iota, z), y, \iota) - V(s, x, y, \iota) \right. \\
& \left. - \mathbf{1}_{\{|z|<1\}}(z) \frac{\partial V(s, x, y, \iota)}{\partial x} \cdot \gamma(s, x, \iota, z) \right) \nu(dz) \\
& \left. + L_2(s, x, y, u_1^*, u_2, \iota) + QV(s, x, y, \cdot)(\iota) \right). \tag{8}
\end{aligned}$$

The corresponding Hamilton Jacobi Isaacs equations of this noncooperative game are

$$\begin{cases} \frac{\partial V_1}{\partial s} = H_1(s, x, y, \iota, V_1, \frac{\partial V_1}{\partial x}, \frac{\partial V_1}{\partial y}, \frac{\partial^2 V_1}{\partial x^2}) \\ \frac{\partial V_2}{\partial s} = H_2(s, x, y, \iota, V_2, \frac{\partial V_2}{\partial x}, \frac{\partial V_2}{\partial y}, \frac{\partial^2 V_2}{\partial x^2}) \\ V_1(T, x, y, \alpha(T)) = \Phi_1(T, x, y, \alpha(T)) \\ V_2(T, x, y, \alpha(T)) = \Phi_2(T, x, y, \alpha(T)). \end{cases} \tag{9}$$

We define  $\tilde{\mu}_1 := \mu(s, x, u_1, u_2^*, \iota)$ ,  $\tilde{L}_1 := L_1(s, x, y, u_1, u_2^*, \iota)$ ,  $\tilde{\mu}_2 := \mu(s, x, u_1^*, u_2, \iota)$ ,  $\tilde{L}_2 := L_2(s, x, y, u_1^*, u_2, \iota)$ , we have the following result.

**Theorem 3.2** Assume that there exists  $(u_1^*, u_2^*) \in \mathcal{U}_1 \times \mathcal{U}_2$  such that the nonlinear Hamilton Jacobi Isaacs equations (9) have solutions  $V_i(s, x, y, \iota)$ ,  $i = 1, 2$ ,

$$u_i^* = \arg \max \left( \tilde{\mu}_i \frac{\partial V_i}{\partial x} - u_1 \frac{\partial V_i}{\partial y} + \tilde{L}_i \right), \quad i = 1, 2. \tag{10}$$

Then the pair  $(u_1^*, u_2^*)$  is a Nash equilibrium solution and  $J_i(s, x, y, \iota; u_1^*, u_2^*) = V_i(s, x, y, \iota)$ ,  $i = 1, 2$ .

**Proof** The proof relies on the fact that this problem can be uncoupled and solved as an optimal control problem. In fact, if we replace the control process  $u_2(\cdot)$  by  $u_2^*(\cdot)$  in (2) then the differential game problem becomes an optimal control problem with the only control variable  $u_1(\cdot)$  the HJB equation of this new

control problem is in fact

$$\begin{cases} \frac{\partial W_1}{\partial s} = H_1(s, x, y, \iota, W_1, \frac{\partial W_1}{\partial x}, \frac{\partial W_1}{\partial y}, \frac{\partial^2 W_1}{\partial x^2}) \\ W_1(T, x, y, \alpha(T)) = \Phi_1(T, x, y, \alpha(T)). \end{cases} \quad (11)$$

Following the assumptions of the Theorem, it is clear that the HJB equation (11) has a solution  $V_1$  and the optimal policy of this new control problem is  $u_1^*$ . Therefore  $u_1^*$  is in equilibrium with  $u_2^*$  and  $V_1$  is the value function of Player 1. A similar argument can be used to show that  $u_2^*$  is in equilibrium with  $u_1^*$  and that  $V_2$  is the value function of Player 2.  $\square$

## 4 Value Function Characterization

In order to solve (9) we will use the notion of viscosity solutions introduced by Crandall and Lions (1983). Let us first recall the definition of viscosity solution.

**Definition 4.1** *The function  $W_i$  defined on  $\mathcal{D} := [0, T] \times \mathbb{R} \times [0, K] \times \mathcal{M}$  is a viscosity subsolution (resp. supersolution) of*

$$\frac{\partial W}{\partial s} = H_i(s, x, y, \iota, W, \frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial^2 W}{\partial x^2}), \quad i = 1, 2 \quad (12)$$

if  $W$  is lower semi-continuous (resp. upper semi-continuous), and for any  $\iota \in \mathcal{M}$ , for any test function  $\phi \in C^{1,2,1}([0, T] \times \mathbb{R} \times [0, K])$  such that  $W - \phi$  has a local maximum (resp. minimum) at  $(s_0, x_0, y_0, \iota) \in \mathcal{D}$

$$\begin{aligned} & \frac{\partial \phi}{\partial s}(s_0, x_0, y_0) \\ & \leq H_i(s_0, x_0, y_0, \iota, W, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial^2 \phi}{\partial x^2}), \end{aligned} \quad (13)$$

$$\begin{aligned} & \left( \text{resp.} \quad \frac{\partial \phi}{\partial s}(s_0, x_0, y_0) \right. \\ & \left. \geq H_i(s_0, x_0, y_0, \iota, W, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial^2 \phi}{\partial x^2}) \right). \end{aligned} \quad (14)$$

**Lemma 4.2** *For each  $\iota \in \mathcal{M}$ , and  $i = 1, 2$  the value function  $V_i(s, x, y, \iota)$  is Lipschitz continuous with respect to  $s, x$  and  $y$ . Moreover, for each  $i = 1, 2$ ,  $V_i$  has at most a linear growth rate, i.e., there exists a constant  $C$  such that  $|V_i(s, x, y, \iota)| \leq C(1 + |x| + |y|)$ .*

The continuity of the value functions  $V_i, i = 1, 2$  naturally comes from the application of the Itô-Lévy isometry, the Lipschitz continuity of the parameters of the model and the Gronwall's inequality. For more details, one can refer to Pemy (2014) for the proof of a similar result in the case of the optimal stopping of regime switching Lévy processes. Below we only proof the linear growth property.

**Proof** The linear growth inequality follows from the Lipschitz continuity of the value functions  $V_i, i = 1, 2$  with respect to  $x$  and  $y$ . Thus there exist  $K, C > 0$  such that

$$|V_i(s, x, y, \iota)| \leq K|x| + |V_i(s, 0, y, \iota)|,$$

and

$$|V_i(s, 0, y, \iota)| \leq C|y| + |V_i(s, 0, 0, \iota)|.$$

Combining the last two inequalities gives,

$$\begin{aligned} |V_i(s, x, y, \iota)| &\leq \max(K, C)(|x| + |y| + |V_i(s, 0, 0, \iota)|) \\ &\leq C'(|x| + |y| + 1), \end{aligned}$$

for some  $C' > \max(K, C)$ . This completes the proof.  $\square$

**Remark 4.3** *The dynamical Programming Principle implies that*

$$\begin{aligned} &V_1(s, x, y, \iota) \\ = &\sup_{u \in \mathcal{U}_1} E \left[ \int_s^T e^{-r(t-s)} L_1(t, X(t), Y(t), u(t), u_2^*(t), \alpha(t)) dt \right. \\ &\left. + e^{-r(T-s)} V_1(T, X(T), Y(T), \alpha(T)) \right] \Bigg| X(s) = x, \\ &Y(s) = y, \alpha(s) = \iota \Bigg] \end{aligned} \quad (15)$$

similarly we have

$$\begin{aligned} &V_2(s, x, y, \iota) \\ = &\sup_{u \in \mathcal{U}_2} E \left[ \int_s^T e^{-r(t-s)} L_2(t, X(t), Y(t), u_1^*(t), u_2(t), \alpha(t)) dt \right. \\ &\left. + e^{-r(T-s)} V_2(T, X(T), Y(T), \alpha(T)) \right] \Bigg| X(s) = x, \\ &Y(s) = y, \alpha(s) = \iota \Bigg]. \end{aligned} \quad (16)$$

Using the classical Dynamical Programming Principle and the continuity of the value functions  $V_1$  and  $V_2$  we can now show that  $V_1$  and  $V_2$  are viscosity solutions of the Hamilton Jacoby Isaacs equations (9).

**Theorem 4.4** *The values functions  $V_i, i = 1, 2$  of the noncooperative game are respectively viscosity solutions of the Isaacs equations (9)*

$$\frac{\partial W}{\partial s} = H_1(s, x, y, \iota, W, \frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial^2 W}{\partial x^2}), \quad (17)$$

$$\frac{\partial W}{\partial s} = H_2(s, x, y, \iota, W, \frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial^2 W}{\partial x^2}). \quad (18)$$

**Proof** We will prove the viscosity solution property of  $V_1$ , a similar argument can be used for  $V_2$ . Let  $\iota \in \mathcal{M}$ , and  $\psi \in C^{1,2,1}([0, T] \times \mathbb{R} \times [0, K])$  such that  $V_1(s, x, y, \iota) - \psi(s, x, y)$  has local minimum at  $(s_0, x_0, y_0)$  in a neighborhood  $N(s_0, x_0, y_0)$ . We set  $\alpha(s_0) = \alpha_0$ , without loss of generality we assume that  $V_1(s_0, x_0, y_0, \alpha_0) - \psi(s_0, x_0, y_0) = 0$  and define a function

$$\varphi(s, x, y, \iota) = \begin{cases} \psi(s, x, y), & \text{if } \iota = \alpha_0, \\ V_1(s, x, y, \iota), & \text{if } \iota \neq \alpha_0. \end{cases} \quad (19)$$

Let  $\gamma \geq s_0$  be the first jump time of  $\alpha(\cdot)$  from the initial state  $\alpha(s_0) = \alpha_0$ , and let  $\eta \in [s_0, \gamma]$  be such that  $(t, X(t), Y(t))$  starts at  $(s_0, x_0, y_0)$  and stays in  $N(s_0, x_0, y_0)$  for  $s_0 \leq t \leq \eta$ . Moreover,  $\alpha(t) = \alpha_0$ , for  $s_0 \leq t \leq \eta$ . Let  $u_1(\cdot)$  be an admissible control such that  $u_1(t) = u$  for  $t \in [0, \eta]$ . From the Dynamical Programming Principle (15) we derive

$$\begin{aligned} & V_1(s_0, x_0, y_0, \alpha_0) \\ \geq & E \left[ \int_{s_0}^{\eta} e^{-r(t-s_0)} L_1 \left( t, X(t), Y(t), u(t), u_2^*(t), \right. \right. \\ & \left. \left. \alpha(t) \right) dt + e^{-r(\eta-s_0)} V_1(\eta, X(\eta), Y(\eta), \alpha(\eta)) \right]. \end{aligned} \quad (20)$$

Using Dynkin's formula we have,

$$\begin{aligned} & E^{s_0, x_0, y_0, \alpha_0} [e^{-r(\eta-s_0)} \varphi(\eta, X(\eta), Y(\eta), \alpha_0)] \\ & - \varphi(s_0, x_0, y_0, \alpha_0) \\ = & E^{s_0, x_0, y_0, \alpha_0} \int_{s_0}^{\eta} e^{-r(t-s_0)} [-r\varphi(t, X(t), Y(t), \alpha_0) \\ & + \mathcal{L}^{u_1, u_2}(\varphi)(t, X(t), Y(t), \alpha_0)] dt. \end{aligned} \quad (21)$$

where  $\mathcal{L}^{u_1, u_2}$  is the generator of the Markov process  $(X_t, Y_t, \alpha(t))$ . Note that  $\mathcal{L}^{u_1, u_2}$  can be written as  $\mathcal{L}^{u_1, u_2}(\varphi)(s, x, y, \iota) = \mathcal{A}^{\iota, u_1, u_2}(\psi)(s, x, y) + Q\varphi(s, x, y, \cdot)(\iota)$  with

$$\begin{aligned} & \mathcal{A}^{\iota, u_1, u_2}(\psi)(s, x, y) \\ = & \frac{\partial \psi}{\partial s} + \frac{1}{2} \sigma^2(s, x, y, \iota) \frac{\partial^2 \psi}{\partial x^2} + \mu(s, x, u_1, u_2, \iota) \frac{\partial \psi}{\partial x} \\ & - u_1 \frac{\partial \psi}{\partial y} + \int_{\mathbb{R}} \left( \psi(s, x + \gamma(s, x, \iota, z), y) - \psi(s, x, y) \right. \\ & \left. - \mathbf{1}_{\{|z| < 1\}}(z) \frac{\partial \psi(s, x, y)}{\partial x} \cdot \gamma(s, x, \iota, z) \right) \nu(dz). \end{aligned}$$

Given that  $(s_0, x_0, y_0)$  is the minimum of  $V_1(t, x, y, \alpha_0) - \psi(t, x, y)$  in  $N(s_0, x_0, y_0)$ . For  $s_0 \leq t \leq \eta$ , we have

$$\begin{aligned} V_1(t, X(t), Y(t), \alpha_0) & \geq \psi(t, X(t), Y(t)) + V_1(s_0, x_0, y_0, \alpha_0) \\ & - \psi(s_0, x_0, y_0) = \varphi(t, X(t), Y(t), \alpha_0). \end{aligned} \quad (22)$$

Using equation (19) and (22), we have

$$\begin{aligned} & E^{s_0, x_0, y_0, \alpha_0} [e^{-r(\eta-s_0)} V_1(\eta, X(\eta), Y(\eta), \alpha_0)] \\ & - V_1(s_0, x_0, y_0, \alpha_0) \\ \geq & E^{s_0, x_0, y_0, \alpha_0} \int_{s_0}^{\eta} e^{-r(t-s_0)} \left[ \mathcal{A}^{\alpha_{s_0}, u_1, u_2^*}(\psi)(t, X(t), Y(t)) \right. \\ & \left. + Q\varphi(t, X(t), Y(t), \cdot)(\alpha_0) - rV_1(t, X(t), Y(t), \alpha_0) \right] dt. \end{aligned} \quad (23)$$

Moreover, we have

$$\begin{aligned} & Q\varphi(t, X(t), Y(t), \cdot)(\alpha_0) \\ = & \sum_{\beta \neq \alpha_0} q_{\alpha_0 \beta} \left( \varphi(t, X(t), Y(t), \beta) - \varphi(t, X(t), Y(t), \alpha_0) \right) \end{aligned} \quad (24)$$



$$\begin{aligned}
&\geq \sum_{\beta \neq \alpha_0} q_{\alpha_0 \beta} \left( V_1(t, X(t), Y(t), \beta) \right. \\
&\quad \left. - V_1(t, X(t), Y(t), \alpha_0) \right) \\
&\geq QV_1(t, X(t), Y(t), \cdot)(\alpha_0).
\end{aligned}$$

Combining (23) and (24), we have

$$\begin{aligned}
&E^{s_0, x_0, y_0, \alpha_0} e^{-r(\eta-s_0)} [V_1(\eta, X(\eta), Y(\eta), \alpha_0)] \\
&\quad - V_1(s_0, x_0, y_0, \alpha_0) \\
&\geq E^{s_0, x_0, y_0, \alpha_0} \int_{s_0}^{\eta} \left[ \mathcal{A}^{\alpha_{s_0}, u_1, u_2^*}(\psi)(t, X(t), Y(t)) \right. \\
&\quad \left. + QV_1(s_0, x_0, y_0, \cdot)(\alpha_s) \right. \\
&\quad \left. - rV_1(t, X(t), Y(t), \alpha_s) \right] e^{-r(t-s_0)} dt.
\end{aligned} \tag{25}$$

It follows from (20) and (25) that

$$\begin{aligned}
&E^{s_0, x_0, y_0, \alpha_0} \int_{s_0}^{\eta} \left[ \mathcal{A}^{\alpha_0, u_1, u_2^*}(\psi)(t, X(t), Y(t)) \right. \\
&\quad \left. + QV_1(t, X(t), Y(t), \cdot)(\alpha_0) - rV_1(t, X(t), Y(t), \alpha_0) \right. \\
&\quad \left. + L(t, X(t), Y(t), u_1(t), u_2^*(t), \alpha(t)) \right] e^{-r(t-s_0)} dt \leq 0.
\end{aligned}$$

Dividing by  $\theta - s_0 > 0$  and then sending  $\theta \rightarrow s_0$  leads to

$$\begin{aligned}
&-rV_1(s_0, x_0, y_0, \alpha_0) + \mathcal{A}^{\alpha_0, u_1, u_2^*}(\psi)(s_0, x_0, y_0) \\
&\quad + QV_1(s_0, x_0, y_0, \cdot)(\alpha_0) \\
&\quad + L(s_0, x_0, y_0, u_1, u_2^*(s_0), \alpha_0) \leq 0.
\end{aligned} \tag{26}$$

Since this inequality is true for any arbitrary control  $u(t) \equiv u_1 \in [0, \bar{u}_1]$ , then taking the supremum over all values  $u \in U_1 = [0, \bar{u}_1]$  we have

$$\begin{aligned}
&rV(s_0, x_0, y_0, \alpha_0) - \sup_{u \in U_1} \left( \mathcal{A}^{\alpha_0, u, u_2^*}(\psi)(s_0, x_0, y_0) \right. \\
&\quad \left. + QV(s_0, x_0, y_0, \cdot)(\alpha_0) \right. \\
&\quad \left. + L(s_0, x_0, y_0, u, u_2^*(s_0), \alpha_0) \right) \geq 0.
\end{aligned} \tag{27}$$

The inequalities (27) obviously proves that the value function  $V$  is a viscosity supersolution as defined in (14).

Now, let us prove the subsolution inequality (13). We want to show that for each  $\iota \in \mathcal{M}$ ,

$$\begin{aligned}
&rV(s_0, x_0, y_0, \iota) - \sup_{u \in U_1} \left( \mathcal{A}^{\alpha_0, u, u_2^*}(\psi)(s_0, x_0, y_0) \right. \\
&\quad \left. + L_1(s_0, x_0, y_0, u, u_2^*(s_0), \iota) \right. \\
&\quad \left. + QV(s_0, x_0, y_0, \cdot)(\iota) \right) \leq 0,
\end{aligned} \tag{28}$$

where  $(s_0, x_0, y_0)$  is a local maximum of  $V_1(s, x, y, \iota) - \psi(s, x, y)$ . Let us assume otherwise that the inequality (28) does not hold. In other terms, we assume that we can find a state  $\alpha_0 \in \mathcal{M}$ , values  $(s_0, x_0, y_0)$  and a function  $\phi \in \mathcal{C}^{1,2,1}([0, T] \times \mathbb{R} \times [0, K])$  such that  $V_1(t, x, y, \alpha_0) - \phi(t, x, y)$  has a local maximum at  $(s_0, x_0, y_0) \in [0, T] \times \mathbb{R} \times [0, K]$ , and we have

$$\begin{aligned} & rV_1(s_0, x_0, y_0, \alpha_0) - \sup_{u \in U_1} \left( \mathcal{A}^{\alpha_0, u, u_2^*}(\psi)(s_0, x_0, y_0) \right. \\ & \quad \left. + QV(s_0, x_0, y_0, \cdot)(\alpha_0) \right. \\ & \quad \left. + L_1(s_0, x_0, y_0, u, u_2^*(s_0), \alpha_0) \right) \geq \delta. \end{aligned} \quad (29)$$

for some constant  $\delta > 0$ .

Let us assume without loss of generality that  $V_1(s_0, x_0, y_0, \alpha_0) - \phi(s_0, x_0, y_0) = 0$ . We define

$$\varphi(s, x, y, i) = \begin{cases} \phi(s, x, y), & \text{if } i = \alpha_0, \\ V_1(s, x, y, i), & \text{if } i \neq \alpha_0. \end{cases} \quad (30)$$

Let  $\gamma$  be the first jump time of  $\alpha(\cdot)$  from the state  $\alpha_0$ , and let  $\eta_0 \in [s_0, \gamma]$  be such that  $(t, X(t), Y(t))$  starts at  $(s_0, x_0, y_0)$  and stays in  $N(s_0, x_0, y_0)$  for  $s_0 \leq t \leq \eta_0$ . Since  $\theta_0 \leq \gamma$  we have  $\alpha(t) = \alpha_0$ , for  $s_0 \leq t \leq \eta_0$ . Moreover, since  $V_1(s_0, x_0, y_0, \alpha_0) - \phi(s_0, x_0, y_0) = 0$  and attains its maximum at  $(s_0, x_0, y_0)$  in  $N(s_0, x_0, y_0)$  then for  $s_0 \leq \eta \leq \eta_0$ ,

$$V_1(\eta, X(\eta), Y(\eta), \alpha(\eta)) \leq \phi(\eta, X(\eta), Y(\eta)).$$

Thus, we also have

$$V_1(\eta, X(\eta), Y(\eta), \alpha(\eta)) \leq \varphi(\eta, X(\eta), Y(\eta), \alpha(\eta)), \quad (31)$$

for  $s_0 \leq \eta \leq \eta_0$ . Using the Dynamical Programming Principle (15), it clear that for any admissible control  $u(\cdot)$  and time  $\tau$  such that  $s_0 < \tau \leq \eta_0$ , we have

$$\begin{aligned} & J_1(s_0, x_0, y_0, \alpha_0; u, u_2^*) \\ & \leq E^{s_0, x_0, y_0, \alpha_0} \left[ e^{-r(\tau-s_0)} V_1(\tau, X(\tau), Y(\tau), \alpha(\tau)) \right. \\ & \quad \left. + \int_{s_0}^{\tau} e^{-r(t-s_0)} L_1(t, X(t), Y(t), u(t), u_2^*(t), \alpha(t)) dt \right] \\ & \leq E^{s_0, x_0, y_0, \alpha_0} \left[ e^{-r(\tau-s_0)} \varphi(\tau, X(\tau), Y(\tau), \alpha(\tau)) \right. \\ & \quad \left. + \int_{s_0}^{\tau} e^{-r(t-s_0)} L_1(t, X(t), Y(t), u(t), u_2^*(t), \alpha(t)) dt \right]. \end{aligned}$$

Note that

$$\begin{aligned} & Q\varphi(t, X(t), Y(t), \cdot)(\alpha_0) \\ & = \sum_{\beta \neq \alpha_0} q_{\alpha_0 \beta} (V_1(t, X(t), Y(t), \beta) - \phi(t, X(t), Y(t))) \\ & \leq \sum_{\beta \neq \alpha_0} q_{\alpha_0 \beta} (V_1(t, X(t), Y(t), \beta) - V_1(t, X(t), Y(t), \alpha_0)) \\ & \leq QV_1(t, X(t), Y(t), \cdot)(\alpha_0). \end{aligned} \quad (32)$$

Using the inequality (29) we have

$$\begin{aligned}
& J(s_0, x_0, y_0, \alpha_0; u, u_2^*) \\
& \leq E^{s_0, x_0, y_0, \alpha_0} \left( \int_{s_0}^{\tau} \left\{ rV(t, X(t), Y(t), \alpha_0) \right. \right. \\
& \quad - \delta - \mathcal{A}^{\alpha_0, u, u_2^*}(\phi)(t, X(t), Y(t)) \\
& \quad \left. \left. - QV(t, X(t), Y(t), \cdot)(\alpha_0) \right\} e^{-r(t-s_0)} dt \right. \\
& \quad \left. + e^{-r(\tau-s_0)} \varphi(\tau, X(\tau), Y(\tau), \alpha_0) \right). \tag{33}
\end{aligned}$$

The Dynkin's formula, (30) and (32) imply that

$$\begin{aligned}
& E^{s_0, x_0, y_0, \alpha_0} e^{-r(\tau-s_0)} \varphi(\tau, X(\tau), Y(\tau), \alpha_0) \\
& = E^{s_0, x_0, y_0, \alpha_0} \int_{s_0}^{\tau} \left[ \mathcal{A}^{\alpha_0, u, u_2^*}(\phi)(t, X(t), Y(t)) \right. \\
& \quad + Q\varphi(t, X(t), Y(t), \cdot)(\alpha_0) \\
& \quad \left. - r\varphi(t, X(t), Y(t), \alpha_0) \right] e^{-r(t-s_0)} dt + \varphi(s_0, x_0, y_0, \alpha_0) \\
& \leq E^{s_0, x_0, y_0, \alpha_0} \int_{s_0}^{\tau} e^{-r(t-s_0)} \left[ \mathcal{A}^{\alpha_0, u, u_2^*}(\phi)(t, X(t), Y(t)) \right. \\
& \quad + QV_1(t, X(t), Y(t), \cdot)(\alpha_0) \\
& \quad \left. - rV_1(t, X(t), Y(t), \alpha_0) \right] dt + V_1(s_0, x_0, y_0, \alpha_0). \tag{34}
\end{aligned}$$

Combining (33) and (34) we have

$$\begin{aligned}
& J(s_0, x_0, y_0, \alpha_0; u, u_2^*) \leq V_1(s_0, x_0, y_0, \alpha_0) \\
& \quad + E^{s_0, x_0, y_0, \alpha_0} \left( - \int_{s_0}^{\tau} e^{-r(t-s_0)} \delta dt \right). \tag{35}
\end{aligned}$$

It is easy to see that the quantity  $\gamma = E^{s_0, x_0, y_0, \alpha_0} \left( \int_{s_0}^{\tau} e^{-r(t-s_0)} \delta dt \right) > 0$ , thus taking the supremum over all admissible control  $u(\cdot) \equiv u$  we obtain

$$V(s_0, x_0, y_0, \alpha_0) \leq -\gamma + V(s_0, x_0, y_0, \alpha_0), \tag{36}$$

which is a contradiction. This proves that the inequality (28) is satisfied. Obviously we derive the subsolution inequality (13). Therefore,  $V$  is a viscosity solution of (9).  $\square$

In order to construct numerical schemes to approximate the value functions  $V_1, V_2$  and prove their convergences we need first to prove the uniqueness of the viscosity solution of the Isaacs equations (9).

**Theorem 4.5 (Maximum Principle)** *If  $U_1(s, x, y, \iota), W_1(s, x, y, \iota)$  are continuous with respect to  $s, x, y$  and viscosity subsolution and supersolution of (17) then,*

$$U_1(s, x, y, \iota) \leq W_1(s, x, y, \iota) \quad \text{for all } (s, x, y, \iota) \in \mathcal{D}.$$

*Similarly if  $U_2(s, x, y, \iota), W_2(s, x, y, \iota)$  are continuous with respect to  $s, x, y$  and viscosity subsolution and supersolution of (18) then,*

$$U_2(s, x, y, \iota) \leq W_2(s, x, y, \iota) \quad \text{for all } (s, x, y, \iota) \in \mathcal{D}.$$

The last theorem implies that the value functions  $V_1, V_2$  are the unique viscosity solutions of the Hamilton Jacobi Isaacs equations (9). The proof of this result follows the classical argument for proving the uniqueness of viscosity solutions, for more details one can refer to Pemy (2014) for the proof of uniqueness in the case of optimal stopping of regime switching Lévy processes.

## 5 Numerical Approximation

In this section, we construct a finite difference scheme and show that it converges to the unique viscosity solutions of the Isaacs equation (9). We will use the following notations; we set  $u = (u_1, u_2)$ ,  $u^* = (u_1^*, u_2^*)$ ,

$$\begin{aligned} V(s, x, y, i) &= \begin{pmatrix} V_1(s, x, y, i) \\ V_2(s, x, y, i) \end{pmatrix}, \\ QV(s, x, y, \cdot)(i) &= \begin{pmatrix} QV_1(s, x, y, \cdot)(i) \\ QV_2(s, x, y, \cdot)(i) \end{pmatrix}, \\ H(s, x, y, i, V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial x^2}) &= \begin{pmatrix} H_1(s, x, y, i, V_1, \frac{\partial V_1}{\partial x}, \frac{\partial V_1}{\partial y}, \frac{\partial^2 V_1}{\partial x^2}) \\ H_2(s, x, y, i, V_2, \frac{\partial V_2}{\partial x}, \frac{\partial V_2}{\partial y}, \frac{\partial^2 V_2}{\partial x^2}) \end{pmatrix}, \\ L(s, x, y, u_1, u_2, i) &= \begin{pmatrix} L_1(s, x, y, u_1, u_2^*, i) \\ L_2(s, x, y, u_1^*, u_2, i) \end{pmatrix}, \end{aligned}$$

and

$$\Phi(s, x, y, i) = \begin{pmatrix} \Phi_1(s, x, y, i) \\ \Phi_2(s, x, y, i) \end{pmatrix},$$

The Isaacs equation (9) can be rewritten as follows

$$\begin{cases} \frac{\partial V}{\partial s} = H(s, x, y, \iota, V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial^2 V}{\partial x^2}), \\ V(T, x, y, \iota) = \Phi(T, x, y, \iota). \end{cases} \quad (37)$$

Let  $k, h, l \in (0, 1)$  be the step size with respect to  $s, x$  and  $y$  respectively, we consider the finite difference operators  $\Delta_s, \Delta_x, \Delta_{xx}$  and  $\Delta_y$  defined by

$$\begin{aligned} \Delta_s V(s, x, y, i) &= \frac{V(s+k, x, y, i) - V(s, x, y, i)}{k}, \\ \Delta_x V(s, x, y, i) &= \frac{V(s, x+h, y, i) - V(s, x, y, i)}{h}, \\ \Delta_y V(s, x, y, i) &= \frac{V(s, x, y+l, i) - V(s, x, y, i)}{l}, \\ \Delta_{xx} V(s, x, y, i) &= \frac{V(s, x+h, y, i) + V(s, x-h, y, i) - 2V(s, x, y, i)}{h^2}. \end{aligned}$$

Let  $If$  denote the integral part of the Hamiltonians  $H_1$  and  $H_2$ . We will approximate  $If$  using the Simpson quadrature. In fact we have

$$\begin{aligned} If(s, x, y, i) &= \int_{\mathbb{R}} \left( f(s, x + \gamma(s, x, i, z), y, i) - f(s, x, y, i) \right. \\ &\quad \left. - \mathbf{1}_{\{|z| < 1\}}(z) \frac{\partial f(s, x, y, i)}{\partial x} \cdot \gamma(s, x, i, z) \right) \nu(dz). \end{aligned}$$

Using the fact the Lévy measure is finite  $\Gamma = \int_{\mathbb{R}} \nu(dz) < \infty$ , we have

$$\begin{aligned} If(s, x, y, i) &= \int_{\mathbb{R}} f(s, x + \gamma(s, x, i, z), y, i) \nu(dz) \\ &\quad - \frac{\partial f(s, x, y, i)}{\partial x} \int_{-1}^1 \gamma(s, x, i, z) \nu(dz) - f(s, x, y, i) \Gamma. \end{aligned}$$

We use the Simpson's quadrature to approximate the integral part of the Hamiltonians. Let  $\xi \in (0, 1)$  the step size of the Simpson's quadrature, the corresponding approximation of the the integral part is

$$\begin{aligned} I_{\xi} f(s, x, y, i) &= \sum_{j=0}^{N_{\xi}} c_j f(s, x + \gamma(s, x, i, z_j), y, i) \\ &\quad - \frac{\partial f(s, x, y, i)}{\partial x} \sum_{j=0}^{M_{\xi}} d_j \gamma(s, x, i, z_j) - f(s, x, y, i) \Gamma, \end{aligned}$$

where the  $(c_j)_{0 \leq j \leq N_{\xi}}$  and  $(d_j)_{0 \leq j \leq M_{\xi}}$  are the corresponding sequences of the coefficients of the Simpson's quadrature. In fact  $\lim_{N_{\xi} \rightarrow \infty} \sum_{j=0}^{N_{\xi}} c_j = \Gamma$  and  $\lim_{M_{\xi} \rightarrow \infty} \sum_{j=0}^{M_{\xi}} d_j = \int_{-1}^1 \nu(dz)$ . The corresponding discrete versions of the Hamiltonians  $H_1, H_2$  are given by

$$\begin{aligned} &H_{1, u_2^*}^{h, k, l} V_1(s, x, y, i) \\ &= rV_1(s, x, y, i) - \sup_{u_1 \in U_1} \left( I_{\xi} V_1(s, x, y, i) \right. \\ &\quad + \frac{1}{2} \sigma^2(s, x, i) \Delta_{xx} V_1(s, x, y, i) \\ &\quad + \mu(s, x, i, u_1, u_2^*) \Delta_x V_1(s, x, y, i) + \\ &\quad - u_1 \Delta_y V_1(s, x, y, i) + L_1(s, x, y, u_1, u_2^*, i) \\ &\quad \left. + QV_1(s, x, y, \cdot)(i) \right), \end{aligned} \tag{38}$$

and

$$\begin{aligned} &H_{2, u_1^*}^{h, k, l} V_2(s, x, y, i) \\ &= rV_2(s, x, y, i) - \sup_{u_2 \in U_2} \left( I_{\xi} V_2(s, x, y, i) \right. \\ &\quad + \frac{1}{2} \sigma^2(s, x, i) \Delta_{xx} V_2(s, x, y, i) \end{aligned}$$

$$\begin{aligned}
& +\mu(s, x, i, u_1^*, u_2)\Delta_x V_2(s, x, y, i) \\
& -u_1^*\Delta_y V_2(s, x, y, i) + L_1(s, x, y, u_1^*, u_2, i) \\
& +QV_2(s, x, y, \cdot)(i) \Big). \tag{39}
\end{aligned}$$

Therefore the discrete version of (37) is

$$\begin{cases} V(s, x, y, i) = \frac{1}{r}\Delta_s V + \frac{1}{r}H_{u_1^*, u_2^*}^{j, k, l} V(s, x, y, i), \\ V(T, x, y, i) = \Phi(T, x, y, i), \end{cases} \tag{40}$$

with

$$H_{u_1^*, u_2^*}^{j, k, l} V(s, x, y, i) = \begin{pmatrix} H_{1, u_2^*}^{h, k, l} V_1(s, x, y, i) - rV_1 \\ H_{2, u_1^*}^{h, k, l} V_2(s, x, y, i) - rV_2 \end{pmatrix}.$$

First we prove the existence of a solution for the discretized equation (40) on bounded subsets of the domain of study  $\mathcal{D}$  where  $\mathcal{D} := [0, T] \times \mathbb{R} \times [0, K] \times \mathcal{M}$ . We define  $\mathcal{D}_R = \{(s, x, y, i) \in \mathcal{D}; |x| < R\}$ , for some  $R > 0$ . We will restrict our study on the set  $\mathcal{D}_R$  for some  $R > 0$  large enough. By doing this we are just assuming that the commodity price will not go beyond a reasonable large threshold. We will approximate our solution on that bounded domain. We have the following crucial Lemma.

**Lemma 5.1** *Let  $\xi > 0$  be small enough, for each  $h, k, l \in (0, 1)$ , there exists a unique bounded function  $V_{l, h, k}$  defined on  $\mathcal{D}_R$  that solves equation (40).*

**Proof.** We define the operator  $\mathcal{F}_\xi$  on bounded functions on  $\mathcal{D}_R$  as follows

$$\begin{aligned}
& \mathcal{F}_\xi(V)(s, x, y, i; h, k, l) \\
& = \frac{1}{r}\Delta_s V + \frac{1}{r}H_{u_1^*, u_2^*}^{j, k, l} V(s, x, y, i) \\
& = \frac{1}{rk}V(s+k, x, y, i) + \sup_{(u_1, u_2) \in U} \left( a_{u, u^*}(s, x, i)V(s, x+h, y, i) \right. \\
& \quad + b(s, x, i)V(s, x-h, y, i) - c_{u, u^*}(s, x, i)V(s, x, y, i) \\
& \quad - \frac{1}{rl}\hat{u}_1 V(s, x, y+l, i) + \sum_{j=0}^{N_\xi} \frac{c_j}{r} V(s, x+\gamma(s, x, i, z_j), y, i) \\
& \quad + \frac{1}{r}L(s, x, y, u_1, u_2, i) + \sum_{j \neq i} \frac{q_{ij}}{r} V(s, x, y, j) \\
& \quad \left. - V(s, x+h, y, i) \frac{\sum_{j=0}^{M_\xi} d_j \gamma(s, x, i, z_j)}{rh} \right), \tag{41} \\
& \mathcal{F}_\xi(V)(T, x, y, i; h, k, l) = \Phi(T, x, y, i).
\end{aligned}$$

Where the coefficients  $\hat{u}_1$ ,  $a_{u, u^*}(s, x, i)$ ,  $b(s, x, i)$  and  $c_{u, u^*}(s, x, i)$  are defined as follows

$$c_{u, u^*}(s, x, i) = \begin{pmatrix} \frac{1}{rk} + \frac{\sigma^2(s, x, i)}{rh^2} + \frac{\mu(s, x, u_1, u_2^*)}{rh} \\ - \frac{\sum_{j=0}^{M_\xi} d_j \gamma(s, x, i, z_j)}{rh} - \frac{u_1}{rl} + \frac{\Gamma}{r} + \sum_{j \neq i} \frac{q_{ij}}{r} \\ \frac{1}{rk} + \frac{\sigma^2(s, x, i)}{rh^2} + \frac{\mu(s, x, u_1^*, u_2)}{rh} \\ - \frac{\sum_{j=0}^{M_\xi} d_j \gamma(s, x, i, z_j)}{rh} - \frac{u_1^*}{rl} + \frac{\Gamma}{r} + \sum_{j \neq i} \frac{q_{ij}}{r} \end{pmatrix},$$

$$a_{u,u^*}(s, x, i) = \left( \frac{\sigma^2(s, x, i)}{2rh^2} + \frac{\mu(s, x, i, u_1, u_2^*)}{rh} \right),$$

$$b(s, x, i) = \frac{\sigma^2(s, x, i)}{2rh^2}, \quad \hat{u}_1 = \begin{pmatrix} u_1 & 0 \\ 0 & u_1^* \end{pmatrix}.$$

Note that equation (40) is equivalent to  $V(s, x, y, i) = \mathcal{F}_\xi(V)(s, x, y, i; h, k, l)$ , it suffices to show the operator  $\mathcal{F}_\xi$  has a fixed point. Using the fact that the difference of sups is less than the sup of differences. If we have two bounded functions  $V, W$  defined on  $\mathcal{D}_R$ , it is clear that

$$\begin{aligned} & |\mathcal{F}_\xi(V)(s, x, y, i; h, k, l) - \mathcal{F}_\xi(W)(s, x, y, i; h, k, l)| \\ & \leq \left| \sup_{u \in U} \left[ \left( a_{u,u^*}(s, x, i) + b(s, x, i) - c_{u,u^*}(s, x, i) \right. \right. \right. \\ & \quad \left. \left. \left. + \frac{1}{rk} + \sum_{j=0}^{N_\xi} \frac{c_j}{r} + \sum_{j \neq i} \frac{q_{ij}}{r} - \frac{1}{rl} \hat{u}_1 \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{\sum_{j=0}^{M_\xi} d_j \gamma(s, x, y, i) z_j}{rh} \right) \sup_{\mathcal{D}_R} |V - W| \right] \right| \\ & \leq \left| \sum_{j=0}^{N_\xi} \frac{c_j}{r} - \frac{\Gamma}{r} \right| \sup_{\mathcal{D}_R} |V - W|. \end{aligned}$$

Therefore, for  $\xi \in (0, 1)$  small enough so that  $\left| \sum_{j=0}^{N_\xi} \frac{c_j}{r} - \frac{\Gamma}{r} \right| < 1$ , the map  $\mathcal{F}_\xi$  is a contraction on the space of bounded functions on  $\mathcal{D}_R$ , using the Banach's Fixed Point Theorem we conclude the proof of the lemma.  $\square$

**Remark 5.2** 1. Define  $S \rightarrow (0, 1)^4 \times [0, T] \times \mathbb{R} \times [0, K] \times \mathcal{M} \times \mathbb{R} \times B([0, T] \times \mathbb{R} \times [0, K] \times \mathcal{M})$  as follows;

$$\begin{aligned} & S(\xi, h, k, l, s, x, y, i, w, W) \\ & = w - \sup_{u \in U} \left( a_{u,u^*}(s, x, i) W(s, x + h, y, i) \right. \\ & \quad + b(s, x, i) W(s, x - h, y, i) - c_{u,u^*}(s, x, i) w \\ & \quad - \frac{1}{rl} \hat{u}_1 W(s, x, y + l, i) + \sum_{j \neq i} \frac{q_{ij}}{r} W(s, x, y, j) \\ & \quad + \sum_{j=0}^{N_\xi} \frac{c_j}{r} W(s, x + \gamma(s, x, i, z_j), y, i) \\ & \quad \left. - W(s, x + h, y, i) \frac{\sum_{j=0}^{M_\xi} d_j \gamma(s, x, i, z_j)}{rh} \right. \\ & \quad \left. + \frac{1}{r} L(s, x, y, u, i) \right), \end{aligned} \tag{42}$$

where coefficients  $\hat{u}_1$ ,  $c_{u,u^*}(s, x, i)$ ,  $a_{u,u^*}(s, x, i)$  and  $b(s, x, i)$  are defined as in Lemma 5.1. Obviously  $V_{h,k,l}$  solves the equation

$$S(\xi, h, k, l, s, x, y, i, V_{h,k,l}(s, x, y, i), V_{h,k,l}) = 0.$$

It is clear that for  $h$  small enough the coefficients of  $a_{u,u^*}(s, x, i)$  are positive moreover,  $b(s, x, i) > 0$  therefore the scheme  $S$  is monotone with respect to argument  $W$  i.e., for all  $\xi, k, l \in (0, 1), s \in [0, T], x \in \mathbb{R}^+, y \in [0, K], i \in \mathcal{M}$  and  $W_1, W_2 \in B([0, T] \times \mathbb{R}^+ \times [0, K] \times \mathcal{M})$  and  $h$  small enough, we have

$$\begin{aligned} & S(\xi, h, k, l, s, x, y, i, w, W_2) \\ & \leq S(\xi, h, k, l, s, x, y, i, w, W_1) \\ & \text{whenever } W_1 \leq W_2. \end{aligned} \tag{43}$$

2. It is clear from Lemma 5.1 that the numerical scheme obtained from (40) is stable since the solution of the scheme is bounded independently of the step sizes  $h, k, l \in (0, 1)$  and obviously consistent because as the step sizes  $h, k, l$  go to zero the finite difference operators converge to the actual partial differential operators. We have the following convergence theorem.

**Theorem 5.3** For each  $\xi > 0$  small enough, let  $V_{h,k,l}$  be the solution of the discrete scheme obtained in Lemma 5.1. Then as  $\xi \downarrow 0$  and  $(h, k, l) \rightarrow 0$  the sequence  $V_{h,k,l}$  converges locally uniformly on  $\mathcal{D}_R$  to the unique viscosity solution  $V$  of (9).

**Proof** Define

$$\begin{aligned} V^*(s, x, y, i) &= \limsup_{\theta \rightarrow s, \eta \rightarrow x, \zeta \rightarrow y, k \downarrow 0, h \downarrow 0, l \downarrow 0} V_{k,h,l}(\theta, \eta, \zeta, i), \\ V_*(s, x, y, i) &= \liminf_{\theta \rightarrow s, \eta \rightarrow x, \zeta \rightarrow y, k \downarrow 0, h \downarrow 0, l \downarrow 0} V_{k,h,l}(\theta, \xi, \zeta, i). \end{aligned} \tag{44}$$

We claim that  $V^*$  and  $V_*$  are sub- and supersolutions of (9), respectively. To prove this claim, we only consider the case for  $V^*$ . The argument for that of  $V_*$  is similar. For each  $i \in \mathcal{M}$ , we want to show

$$\frac{\partial \Psi}{\partial s}(s_0, x_0, y_0) \leq H(s_0, x_0, y_0, i, V^*, \frac{\partial \Psi}{\partial x}, \frac{\partial \Psi}{\partial y}, \frac{\partial^2 \Psi}{\partial x^2}),$$

for any test function  $\Psi \in \mathcal{C}^{1,2,1}([0, T] \times \mathbb{R} \times [0, K])$  such that  $(s_0, x_0, y_0, i)$  is a strictly local maximum of  $V^*(s, x, y, i) - \Psi(s, x, y)$ . Without loss of generality, we may assume that  $V^*(s_0, x_0, y_0, i) = \Psi(s_0, x_0, y_0)$  and because of the stability of our scheme we can also assume that  $\Psi \geq \sup_{k,h,l} \|V_{k,h,l}\|$  outside of the ball  $B((s_0, x_0, y_0), r)$  where  $r > 0$  is such that

$$\begin{aligned} & V^*(s, x, y, i) - \Psi(s, x, y) \leq 0 = V^*(s_0, x_0, y_0, i) \\ & - \Psi(s_0, x_0, y_0) \text{ in } B((s_0, x_0, y_0), r). \end{aligned}$$

This implies that there exist sequences  $k_n > 0, h_n > 0, l_n > 0$  and  $(\theta_n, \eta_n, \zeta_n) \in [0, T] \times \mathbb{R}^+ \times [0, K]$  such that as  $n \rightarrow \infty$  we have

$$\begin{aligned} & k_n \rightarrow 0, \quad h_n \rightarrow 0, \quad l_n \rightarrow 0, \quad \theta \rightarrow s_0, \quad \eta_n \rightarrow x_0, \\ & \zeta_n \rightarrow y_0, \quad V_{k_n, h_n, l_n}(\theta_n, \eta_n, \zeta_n, i) \rightarrow V^*(s_0, x_0, y_0, i), \\ & \text{and } (\theta_n, \eta_n, \zeta_n) \text{ is a global maximum of } V_{k_n, h_n, l_n} - \Psi. \end{aligned}$$

Denote  $\epsilon_n = V_{k_n, h_n, l_n}(\theta_n, \eta_n, \zeta_n, i) - \Psi(\theta_n, \eta_n, \zeta_n)$ . Obviously  $\epsilon_n \rightarrow 0$  and

$$V_{k_n, h_n, l_n}(s, x, y, i) \leq \Psi(s, x, y) + \epsilon_n \text{ for all } (s, x, y) \in [0, T] \times \mathbb{R} \times [0, K]. \tag{45}$$



We know that for all  $\xi > 0$ , small enough,

$$S(\xi, k_n, h_n, l_n, \theta_n, \eta_n, \zeta_n, i, V_{k_n, h_n, l_n}(\theta_n, \eta_n, \zeta_n, i), V_{k_n, h_n, l_n}) = 0.$$

The monotonicity of  $S$  and (45) implies

$$\begin{aligned} & S(\xi, k_n, h_n, l_n, \theta_n, \eta_n, \zeta_n, i, \Psi(\theta_n, \eta_n, \zeta_n) + \epsilon_n, \Psi + \epsilon_n) \\ & \leq S(\xi, k_n, h_n, l_n, \theta_n, \eta_n, \zeta_n, i, V_{k_n, h_n, l_n}(\theta_n, \eta_n, \zeta_n, i), V_{k_n, h_n, l_n}) = 0, \end{aligned} \quad (46)$$

when  $n > N$  for some large integer  $N$ . Therefore,

$$\lim_{\xi \downarrow 0} \lim_{n \rightarrow \infty} S(\xi, k_n, h_n, l_n, \theta_n, \eta_n, \zeta_n, i, \Psi(\theta_n, \eta_n, \zeta_n) + \epsilon_n, \Psi + \epsilon_n) \leq 0,$$

so

$$\frac{\partial \Psi}{\partial s}(s_0, x_0, y_0) \leq H(s_0, x_0, y_0, i, V^*, \frac{\partial \Psi}{\partial x}, \frac{\partial \Psi}{\partial y}, \frac{\partial^2 \Psi}{\partial x^2}).$$

This proves that  $V^*$  is a viscosity subsolution and, similarly we can prove that  $V_*$  is a viscosity supersolution. Thus, using the uniqueness of the viscosity solution, we see that  $V = V^* = V_*$ . Therefore, we conclude that the sequence  $(V_{h,k,l})_{h,k,l}$  converges locally uniformly to  $V$  as desired.  $\square$

This result is the standard method for approximating viscosity solutions, for more one can refer to Barles and Souganidis (1991). Below we present a fixed point algorithm that can be used in numerical implementations of this method.

#### *Fixed Point Algorithm*

1. Choose a tolerance  $\epsilon > 0$ . Choose an initial guess for  $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$  denoted by  $V^{(0)} = \begin{pmatrix} V_1^{(0)} \\ V_2^{(0)} \end{pmatrix}$ .
2. For  $j = 0, \dots, \text{MaxIteration}$ .
  - (a) Find  $u_1^*, u_2^*$  such that

$$\begin{aligned} u_1^* = \arg \max_{u_1 \in U_1} & \left( \mu(s, x, u_1, u_2^*, i) \frac{\partial V_1^{(j)}}{\partial x} - u_1 \frac{\partial V_1^{(j)}}{\partial y} \right. \\ & \left. + L_1(s, x, y, u_1, u_2^*, \iota) \right), \end{aligned} \quad (47)$$

$$\begin{aligned} u_2^* = \arg \max_{u_2 \in U_2} & \left( \mu(s, x, u_1^*, u_2, i) \frac{\partial V_1^{(j)}}{\partial x} - u_2 \frac{\partial V_2^{(j)}}{\partial y} \right. \\ & \left. + L_2(s, x, y, u_1^*, u_2, \iota) \right), \end{aligned} \quad (48)$$

- (b) Solve equation (40)

$$V^{(j+1)} = \frac{1}{r} \Delta_s V^{(j)} + \frac{1}{r} H_{u_1^*, u_2^*} V^{(j)},$$

3. If  $\|V^{(j+1)} - V^{(j)}\| < \epsilon$ , then stop, else go to step 2 with  $j \leftarrow j + 1$ .

## 6 Applications

We apply the results obtained the previous sections to some typical real life situations. We consider an agreement where a multinational oil company will 40% of profits and the government takes 60%, so  $\theta = 0.4$ . The oil field has a known capacity of  $K=10$  billion barrels and the lease has a  $T=10$  years maturity. We assume that the outputs may affect the market price of oil so the drift of our diffusion has the form  $\mu(t, x, u_1, u_2, i) = x(\bar{\mu}(i) - \rho u_1)$ . The oil price dynamic is as follows

$$dX(t) = X(t) \left( (\bar{\mu}(\alpha(t)) - \rho u_1(t)) dt + \sigma(\alpha(t)) dW_t + \int_{\mathbb{R}} \gamma(\alpha(t)) z \bar{N}(dt, dz) \right).$$

The parameter  $\rho \in [0, 1)$  will capture the relative impact of the oil production on the oil price. When the market is bullish we have the following parameters  $\alpha(t) = 1$ ,  $\bar{\mu}(1) = 0.2$ ,  $\sigma(1) = 0.2$ ,  $\gamma(1) = 0.2$ . When the market is bearish we have  $\alpha(t) = 2$ ,  $\bar{\mu}(2) = -0.02$ ,  $\sigma(2) = 0.3$ ,  $\gamma(2) = 0.3$ . The generator of the Markov chain is

$$Q = \begin{pmatrix} -0.001 & 0.001 \\ 0.002 & -0.002 \end{pmatrix}.$$

We assume that the profit rate function of the oil company per unit of time (hour) for each barrel of crude oil extracted is

$$L_1(t, x, y, u_1, u_2) = 0.4(xu_1 - 25u_1)(1 - u_2).$$

The terminal profit rate of the oil company is  $\Phi_1(T, x, y) = 0$  because at the end of the contract there are no extractions. The profit rate function of the government is

$$L_2(t, x, y, u_1, u_2) = 0.6(xu_1 - 25u_1) + 0.4u_2(xu_1 - 25u_1).$$

The terminal profit rate of the government is the market value of the remaining reserve,  $\Phi_2(T, x, y) = y(x - 25)$ . Moreover, we assume that the extraction  $u_1(\cdot) \in [0, 50000]$  and the top tax rate is 20% so  $u_2(\cdot) \in [0, 0.2]$ . Keep in mind that, because the payoff rates are linear functions of each control variable  $u_1(\cdot), u_2(\cdot)$ . Once the value functions  $V_1, V_2$  are approximated numerically, using Theorem 3.2 the optimal strategies  $u_1^*, u_2^*$  are obtained by looking at the sign of the derivative of the following quantities with respect to  $u_1$  and then  $u_2$

$$\begin{aligned} & x(\bar{\mu}(i) - \rho u_1) \frac{\partial V_1}{\partial x} - u_1 \frac{\partial V_1}{\partial y} \\ & + 0.4(xu_1 - 25u_1)(1 - u_2) \quad \text{for } u_1^*, \\ & -u_1 \frac{\partial V_2}{\partial y} + 0.6(xu_1 - 25u_1) \\ & + 0.4u_2(xu_1 - 25u_1) \quad \text{for } u_2^*. \end{aligned}$$

Let the functions  $F$  and  $G$  be those respective derivatives,

$$\begin{aligned} F(s, x, y, u_1, u_2, i) &= -x\rho \frac{\partial V_1}{\partial x} - \frac{\partial V_1}{\partial y} \\ &+ 0.4(x - 25)(1 - u_2), \end{aligned}$$

$$G(s, x, y, u_1, u_2, i) = 0.4(xu_1 - 25u_1).$$

We see that the optimal strategies will only be attained at the endpoints of the intervals  $U_1 = [0, 50000]$  and  $U_2 = [0, 0.2]$ , we have.

$$u_1^*(s) = \begin{cases} 0 & \text{if } F(s, x, y, u_1, u_2, i) \leq 0 \\ 50000 & \text{if } F(s, x, y, u_1, u_2, i) > 0, \end{cases}$$

and

$$u_2^*(s) = \begin{cases} 0 & \text{if } G(s, x, y, u_1, u_2, i) \leq 0 \\ 0.2 & \text{if } G(s, x, y, u_1, u_2, i) > 0. \end{cases}$$

In Figure 1 we have plots of the function  $F$  at various times and when the market is bullish and bearish. Note that the sign of this function will dictate our optimal extraction policy. In all these plots, the region above the line represents the domain where it is always optimal for the oil company to extract at full capacity and the region below the curve represents the domain where it is better not to extraction. Moreover, it is easy to see that the function  $G$  is positive when  $x > 25$ . Thus it is optimal for the government to tax the oil company at the full rate of 20% when the oil price is above 25 and not tax the oil company at all when the oil price is below 25.

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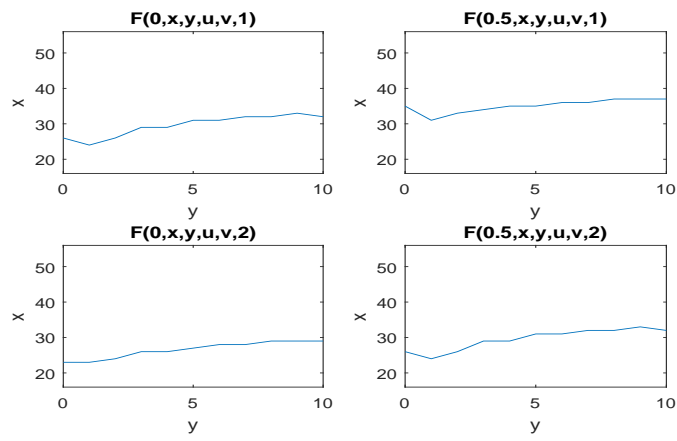


Figure 1: Plots of  $F(s, x, y, u, v, i)$  at times  $s = 0, 0.5$

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